The Maximum Distance Problem and Band Sequences

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ABSTRACT

We solve the following problem. For $1 \leqslant j, k \leqslant n$ and $|j-k| \leqslant m$, let a_{jk} be a given complex number with $a_{kj} = \bar{a}_{jk}$. We wish to find linearly independent vectors $x_1, \dots, x_n$ such that $\langle x_k, x_j \rangle = a_{jk}$ for $|j-k| \leqslant m$ and such that the distance from x_k to the linear span of $x_1, \dots, x_{k-1}$ is maximal for $2 \leqslant k \leqslant n$. We construct and

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characterize all such sequences of vectors. Our solution leads naturally to the class of m-band sequences of vectors in an inner product space. We study these sequences and characterize their equivalence classes under unitary transformations.

1. INTRODUCTION

One of the main results of this paper is the solution of the maximum distance problem. In order to state this problem, we first introduce some notation and terminology.

For any vectors $x_1,\dots,x_n$ in $\mathbf{C}^n$ and for $2 \leqslant k \leqslant n$, let $\mathrm{sp}\langle x_1,\dots,x_k\rangle$ denote the subspace spanned by $x_1,\dots,x_k$ and let $\mathrm{dist}(x_k,\mathrm{sp}\langle x_1,\dots,x_{k-1}\rangle)$ denote the distance from x_k to $\mathrm{sp}\langle x_1,\dots,x_{k-1}\rangle$. Given $0 \leqslant m < n$ and a set $\{a_{jk}: |j-k| \leqslant m\}$ of complex numbers satisfying $a_{kj} = \bar{a}_{jk}$, we shall say that a sequence of vectors $\{x_k\}_{k=1}^n$ is *admissible* if it is linearly independent and if

$$\langle x_k, x_j\rangle = a_{jk} \qquad (|j-k| \leqslant m). \tag{1.1}$$

We define

$$d_k = \sup \mathrm{dist}(x_k,\mathrm{sp}\langle x_1,\dots,x_{k-1}\rangle) \qquad (2 \leqslant k \leqslant n),$$

where the supremum is taken over all admissible sequences of vectors in $\mathbf{C}^n$. The *maximum distance problem* is to describe all admissible sequences $\{x_k\}_{k=1}^n$ such that

$$\mathrm{dist}(x_k,\mathrm{sp}\langle x_1,\dots,x_{k-1}\rangle) = d_k \qquad (2 \leqslant k \leqslant n).$$

Each such sequence is called a solution of the maximum distance problem. This maximum distance problem has close connections with maximum entropy in the mathematical theory of signal processing [3, 5, 6].

For any sequence $\{x_k\}_{k=1}^n$ in $\mathbf{C}^N$, we denote the *Gram matrix* as

$$\Gamma(x_1,\dots,x_n) = (\langle x_k, x_j\rangle)_{j,k=1}^n;$$

it is positive definite if $\{x_k\}_{k=1}^n$ is linearly independent. The constraints in (1.1) simply specify the entries in $\Gamma(x_1,\dots,x_n)$ that lie in a band centered along the main diagonal. Let $R = (a_{jk})$ be the $n \times n$ band matrix obtained by setting $a_{jk} = 0$ for $|j-k| > m$. For any matrix $B = (b_{pq})_{p,q=1}^n$ and for

$1 \leqslant j \leqslant k \leqslant n$, we denote by $B(j,\ldots,k)$ the principal submatrix $(b_{pq})_{p,q=j}^{k}$. Clearly, if the maximum distance problem with constraints (1.1) has a solution, then each submatrix $R(j,\ldots,j+m)$ is positive definite for $1 \leqslant j \leqslant n-m$. Conversely, if $R(j,\ldots,j+m)$ is positive definite for $1 \leqslant j \leqslant n-m$, then the maximum distance problem with constraints (1.1) has a solution. We shall prove this in Section 3, where we also construct all solutions of the maximum distance problem in two different ways—one by a recursion formula and the other by solving a set of systems of linear equations. We shall show that when the problem has a solution, the maximum distances may be calculated directly from the data a_{jk} used in (1.1) to define the problem. In fact, for $2 \leqslant k \leqslant n$, the maximum distance d_k equals

$$\Delta_k = \sqrt{\frac{\det R(\beta(k),\ldots,k)}{\det R(\beta(k),\ldots,k-1)}}$$

where $\beta(k) = \max\{1, k-m\}$.

The study of the maximum distance problem leads naturally to the notion of a band sequence of vectors in $\mathbf{C}^N$. If we apply the Gram-Schmidt process to a sequence of vectors $\{y_k\}_{k=1}^{n}$, we obtain a sequence $\{\omega_k\}_{k=1}^{n}$ of orthonormal vectors that is related to $\{y_k\}$ by a system of equations that may be written in the form

$$a_{1k}y_1 + \cdots + a_{kk}y_k = \omega_k \qquad (k=1,\ldots,n). \tag{1.2}$$

Definition. Let m and n be integers with $0 \leqslant m < n$. We say that a linearly independent sequence $\{y_k\}_{k=1}^{n}$ is an *m-band sequence* if $a_{jk} = 0$ for $2 \leqslant k \leqslant n$ and $1 \leqslant j < \beta(k)$.

For an m-band sequence, (1.2) may be written in the form

$$a_{\beta(k),k}y_{\beta(k)} + \cdots + a_{kk}y_k = \omega_k \qquad (k=1,\ldots,n). \tag{1.3}$$

Thus, for each k, ω_k may be expressed as a linear combination of at most $m+1$ of the vectors $y_1,\ldots,y_k$.

Vectors in a 0-band sequence are orthogonal, and every linearly independent sequence $\{y_k\}_{k=1}^{n}$ is an $(n-1)$-band sequence. The smallest m for which a sequence is an m-band sequence may be considered as a measure of how near the sequence is to being orthogonal.

In Section 2 we characterize m-band sequences in several ways, and then we show in Section 3 that a sequence is a solution of the maximum distance problem if and only if it is an m-band sequence that satisfies the constraints (1.1). In Section 4, we study the equivalence classes of m-band sequences under unitary transformations. It is clear that the m-band in the Gram matrix is the same for every sequence in a class. It turns out that the entries in this band uniquely determine the class. Furthermore, we show that the set of all solutions of the maximum distance problem is such an equivalence class of m-band sequences determined by the band specified in (1.1).

In Section 5 we consider a maximum volume problem and show that it is equivalent to the maximum distance problem. Then we consider a special case of the maximum distance problem in which the solution is required to be composed of two orthonormal sequences.

2. BAND SEQUENCES

It will be useful to describe m-band sequences in terms of matrices. We suppose that $\{y_k\}_{k=1}^n$ is a linearly independent sequence in an N-dimensional inner product space, where $N \geqslant n$. Identify the y_k and the ω_k with vectors in $\mathbf{C}^N$, and construct matrices $Y = [y_1 \ \cdots \ y_n]$ and $U = [\omega_1 \ \cdots \ \omega_n]$. Also, let $a_{jk} = 0$ for $1 \leqslant k < j \leqslant n$, and let A be the upper triangular matrix (a_{jk}). Then (1.2) is equivalent to the matrix equation

$$YA = U. \tag{2.1}$$

We call the matrix A that arises from applying the Gram-Schmidt process to $\{y_k\}$ the *orthonormalization matrix* of $\{y_k\}_{k=1}^n$. The condition that makes $\{y_k\}$ an m-band sequence is equivalent to the condition that A is a band matrix with bandwidth m. In general, a matrix (b_{jk}) is a band matrix with bandwidth m if $b_{jk} = 0$ for $|j-k| > m$.

PROPOSITION 2.1. *Let $\{y_k\}_{k=1}^n$ be a linearly independent sequence of vectors in $\mathbf{C}^N$ and let $Y = [y_1 \ \cdots \ y_n]$. Then the following statements are equivalent*:

(1) *$\{y_k\}$ is an m-band sequence.*

(2) *There exists an $n \times n$ invertible upper triangular band matrix B with bandwidth m, and an $N \times n$ matrix V with orthonormal columns, such that $YB = V$.*

(3) *There exists an $n \times n$ invertible lower triangular band matrix C with bandwidth m, and an $N \times n$ matrix W with orthonormal columns, such that $YC = W$.*

(4) *The Gram matrix* $\Gamma(y_1,\dots,y_n) = Y^*Y$ *is invertible, and its inverse is a band matrix with bandwidth* m.

(5) *Every submatrix of* $\Gamma(y_1,\dots,y_n) = \{\gamma_{ij}\}$ *belonging to the part of* $\{\gamma_{ij}\}$ *for which* $j + m > i$ *has rank less than or equal to* m.

Proof. Clearly (2) follows from (1). Suppose that (2) holds, and assume first that $n = N$. Given $YB = V$, let A and U be as in (2.1). Then $Y = UA^{-1} = VB^{-1}$ and hence $B^{-1}A = V^{-1}U$. Since $B^{-1}A$ is upper triangular and $V^{-1}U$ is unitary, $B^{-1}A$ must be a diagonal matrix. Thus $A = B(B^{-1}A)$ is a band matrix with bandwidth m. This argument may be applied to the case $n < N$ by letting $\{y_k\}_{k=n+1}^{N}$ be an orthonormal set that is orthogonal to $\{y_k\}_{k=1}^{n}$ and considering the matrix $[y_1 \ \cdots \ y_N]$. Thus (2) implies (1).

Next, given (2), there exists a lower triangular $n \times n$ matrix C such that $BB^* = CC^*$. Also, C is invertible because B is. Furthermore, $B^{-1}C$ is unitary, because $(BB^*)^{-1} = C^{*-1}C^{-1}$ and $(B^{-1}C)^*(B^{-1}C) = C^*(BB^*)^{-1}C = I$. Thus $YC = V(B^{-1}C)$, and the columns of $VB^{-1}C$ are orthonormal. Hence (2) implies (3).

If (3) holds, then

$$I = (YC)^*(YC) = C^*(Y^*Y)C.$$

Since C is invertible, so is Y^*Y, and $(Y^*Y)^{-1} = CC^*$. Since C is a lower triangular band matrix with bandwidth m, it follows that CC^* is a band matrix with bandwidth m. Thus (3) implies (4).

Next, suppose that (4) holds. Then there exists an invertible upper triangular matrix B such that

$$(Y^*Y)^{-1} = BB^*. \tag{2.2}$$

The relation $(Y^*Y)^{-1}B^{*-1} = B$ implies that B is a band matrix with bandwidth m, because $(Y^*Y)^{-1}$ is of this type, B^{*-1} is lower triangular, and the product $(Y^*Y)^{-1}B^{*-1}$ is upper triangular. Furthermore, from (2.2) we have $Y^*Y = B^{*-1}B^{-1}$, $B^*(Y^*Y)B = I$, and finally $(YB)^*YB = I$, which shows that (2) is implied by (4).

The equivalence of (4) and (5) follows immediately from a theorem of Asplund [1, Theorem 1] and the fact that $\Gamma(y_1,\dots,y_n\}$ is self-adjoint. This completes the proof. ∎

The proof above shows that the matrix B in (2) is unique up to multiplication by a diagonal unitary matrix. The orthonormalization matrix is the particular choice of B whose diagonal entries are positive. The matrix C

in (3) arises when the Gram-Schmidt process is applied to the reverse sequence $y_n, \dots, y_1$.

EXAMPLE. The equation below shows that the columns of the matrix on the left form a 1-band sequence for any complex number α:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha & 1 & 0 & \cdots & 0 \\ \alpha^2 & \alpha & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha^{N-1} & \alpha^{N-2} & \alpha^{N-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha & 1 & 0 & & & 0 \\ 0 & -\alpha & 1 & & & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Statement (2) of Proposition 2.1 says that every band sequence arises as the columns of a matrix UB^{-1}, where the columns of U are orthonormal and B is an upper triangular band matrix. Moreover, if Y is a matrix whose columns form an m_1-band sequence, and if D is an invertible upper triangular band matrix with bandwidth m_2, then the columns of YD^{-1} will form an $(m_1 + m_2)$-band sequence, since $(YD^{-1})(DB) = YB$. Also, right multiplication of Y by BD^{-1} converts Y into a matrix $Y(BD^{-1})$ whose columns form an m_2-band sequence. Similar conclusions follow from (3) of Proposition 2.1 when D is an invertible lower triangular band matrix.

The next two propositions provide other ways of constructing an m-band sequence from a given m-band sequence.

PROPOSITION 2.2. *The sequence $\{y_1, \dots, y_n\}$ is an m-band sequence if and only if $\{y_n, \dots, y_1\}$ is an m-band sequence.*

Proof. This follows from Proposition 2.1 and the fact that $YA = V$ if and only if $(YE)(EAE) = VE$, where E is the matrix obtained by reversing the order of the columns of the $n \times n$ identity matrix. ∎

PROPOSITION 2.3. *Let $\{y_k\}_{k=1}^{n}$ be an m-band sequence. Then the sequence $\{y_k\}_{k=\alpha}^{\beta}$ is an m-band sequence for $1 \leqslant \alpha \leqslant \beta \leqslant n$.*

Proof. Let A and U be as in (2.1). Let Y_β and U_β be the matrices consisting of the first β columns of Y and U, respectively, and let A_β be the $\beta\times\beta$ principal submatrix $A(1,\dots,\beta)$. Since A is upper triangular, it follows that

$$Y_\beta A_\beta = U_\beta.$$

Therefore $\{y_k\}_{k=1}^{\beta}$ is an m-band sequence. Hence, by Proposition 2.2, $\{y_k\}_{k=\beta}^{1}$ is an m-band sequence. But then the first part of the proof implies that $\{y_k\}_{k=\beta}^{\alpha}$ and hence $\{y_k\}_{k=\alpha}^{\beta}$ are also m-band sequences. ∎

Several geometric characterizations of m-band sequences are given in the next proposition. For $k=1,\dots,n$, let $\gamma(k)=\min(k+m,n)$. Recall that $\beta(k)=\max\{1,k-m\}$ for $2\leqslant k\leqslant n$.

PROPOSITION 2.4. *Let $\{y_k\}_{k=1}^{n}$ be a linearly independent sequence of vectors in $\mathbf{C}^N$. Then the following statements are equivalent:*

(1) *$\{y_k\}$ is an m-band sequence.*
(2) *For $2\leqslant k\leqslant n$,* $\operatorname{proj}(y_k,\operatorname{sp}\langle y_1,\dots,y_{k-1}\rangle)$ $=\operatorname{proj}(y_k,\operatorname{sp}\langle y_{\beta(k)},\dots,y_{k-1}\rangle)$.
(3) *For $2\leqslant k\leqslant n$,* $\operatorname{dist}(y_k,\operatorname{sp}\langle y_1,\dots,y_{k-1}\rangle)$ $=\operatorname{dist}(y_k,\operatorname{sp}\langle y_{\beta(k)},\dots,y_{k-1}\rangle)$.
(4) *For $1\leqslant k\leqslant n-1$,* $\operatorname{proj}(y_k,\operatorname{sp}\langle y_{k+1},\dots,y_n\rangle)$ $=\operatorname{proj}(y_k,\operatorname{sp}\langle y_{k+1},\dots,y_{\gamma(k)}\rangle)$.
(5) *For $1\leqslant k\leqslant n-1$,* $\operatorname{dist}(y_k,\operatorname{sp}\langle y_{k+1},\dots,y_n\rangle)$ $=\operatorname{dist}(y_k,\operatorname{sp}\langle y_{k+1},\dots,y_{\gamma(k)})\rangle)$.

Proof. The Gram-Schmidt process (1.2) may also be written in the form

$$a_{11}y_1=\omega_1, \tag{2.3}$$

$$a_{kk}(y_k-u_k)=\omega_k \qquad (2\leqslant k\leqslant n), \tag{2.4}$$

where

$$u_k=\operatorname{proj}(y_k,\operatorname{sp}\langle y_1,\dots,y_{k-1}\rangle), \tag{2.5}$$

$$a_{11}=\langle y_1,y_1\rangle^{-1/2}, \tag{2.6}$$

and

$$a_{kk}=\operatorname{dist}(y_k,\operatorname{sp}\langle y_1,\dots,y_{k-1}\rangle)^{-1}. \tag{2.7}$$

From (1.3) it follows that $\{y_k\}$ is an m-band sequence if and only if for $k=2,\dots,n$, u_k actually lies in the smaller subspace $\mathrm{sp}\langle y_{\beta(k)},\dots,y_{k-1}\rangle$. For each k, this statement about u_k is easily seen to be equivalent to the statement that u_k equals the projection of y_k onto the smaller subspace, and this in turn is equivalent to the statement that the distance of y_k to $\mathrm{sp}\langle y_1,\dots,y_{k-1}\rangle$ equals the distance of y_k to the smaller subspace. Thus (1), (2), and (3) are equivalent statements. The equivalence of (1), (4), and (5) now follows from Proposition 2.2. ∎

3. THE MAXIMUM DISTANCE PROBLEM

The statement of the maximum distance problem was given in the introduction.

THEOREM 3.1. *Let R be an $n\times n$ band matrix with bandwidth m such that $R(j,\dots,j+m)$ is positive definite for $1\leqslant j\leqslant n-m$. Then every admissible m-band sequence $\{y_k\}_{k=1}^n$ is a solution of the maximum distance problem for R. The maximum distances are given by $d_k=\Delta_k$ for $2\leqslant k\leqslant n$.*

Proof. For any sequence $\{x_k\}_{k=1}^n$ that is admissible for R, it follows from [7, p. 14] that

$$\begin{aligned}\operatorname{dist}(x_k,\mathrm{sp}\langle x_{\beta(k)},\dots,x_{k-1}\rangle)&=\sqrt{\frac{\det\Gamma(x_{\beta(k)},\dots,x_k)}{\det\Gamma(x_{\beta(k)},\dots,x_{k-1})}}\\&=\sqrt{\frac{\det R(\beta(k),\dots,k)}{\det R(\beta(k),\dots,k-1)}}\\&=\Delta_k\qquad(2\leqslant k\leqslant n).\end{aligned}\tag{3.1}$$

Since the distance in (3.1) is greater than or equal to $\operatorname{dist}(x_k,\mathrm{sp}\langle x_1,\dots,x_{k-1}\rangle)$, it follows that $\Delta_k\geqslant d_k$ for $2\leqslant k\leqslant n$. However, if $\{y_k\}_{k=1}^n$ is an admissible m-band sequence, then for $2\leqslant k\leqslant n$, the corresponding distance in (3.1) for the sequence $\{y_k\}$ equals $\operatorname{dist}(y_k,\mathrm{sp}\langle y_1,\dots,y_{k-1}\rangle)$. This implies that $\{y_k\}$ is a solution of the maximum distance problem, and $d_k=\Delta_k$ for $2\leqslant k\leqslant n$. ∎

THEOREM 3.2. *Let R be an $n \times n$ band matrix with bandwidth m, and assume that $R(j,\dots,j+m)$ is positive definite for $1 \leqslant j \leqslant n-m$. Let $\{\omega_k\}_{k=1}^{n}$ be an orthonormal sequence in $\mathbf{C}^N$. Define*

$$y_1 = \sqrt{R_{11}}\,\omega_1 \tag{3.2}$$

and

$$y_k = \Delta_k \omega_k - \frac{1}{\det R(\beta(k),\dots,k-1)} \times \det \begin{bmatrix} R_{\beta(k),\beta(k)} & \cdots & R_{\beta(k),k-1} & R_{\beta(k),k} \\ \vdots & & \vdots & \vdots \\ R_{k-1,\beta(k)} & \cdots & R_{k-1,k-1} & R_{k-1,k} \\ y_{\beta(k)} & \cdots & y_{k-1} & 0 \end{bmatrix} \quad (2 \leqslant k \leqslant n). \tag{3.3}$$

Then $\{y_k\}_{k=1}^{n}$ is an m-band sequence in $\mathbf{C}^N$ that is admissible for R. Moreover, every admissible m-band sequence in $\mathbf{C}^N$ may be obtained in this way.

Proof. Since

$$\operatorname{proj}(y_k, \operatorname{sp}\langle y_1,\dots,y_{k-1}\rangle) \in \operatorname{sp}\langle y_{\beta(k)},\dots,y_{k-1}\rangle,$$

it follows from Proposition 2.4 that $\{y_k\}_{k=1}^{n}$ is an m-band sequence. We shall prove by induction that $\{y_k\}$ satisfies the constraints that make it an admissible sequence. Clearly $\langle y_1, y_1\rangle = R_{11}$. Assume that $\langle y_i, y_j\rangle = R_{ji}$ for $1 \leqslant i, j < k \leqslant n$ and $|i-j| \leqslant m$. Let j be any integer with $\beta(k) \leqslant j < k$. Since $\operatorname{sp}\langle y_1,\dots,y_j\rangle = \operatorname{sp}\langle \omega_1,\dots,\omega_j\rangle$, we have $\langle \omega_k, y_j\rangle = 0$. Thus it follows from (3.3) and the inductive hypothesis that

$$\langle y_k, y_j\rangle = \frac{-1}{\det R(\beta(k),\dots,k-1)} \times \det \begin{bmatrix} R_{\beta(k),\beta(k)} & \cdots & R_{\beta(k),k-1} & R_{\beta(k),k} \\ \vdots & & \vdots & \vdots \\ R_{k-1,\beta(k)} & \cdots & R_{k-1,k-1} & R_{k-1,k} \\ R_{j,\beta(k)} & \cdots & R_{j,k-1} & 0 \end{bmatrix}$$

Subtracting the row in the determinant beginning with $R_{j,\beta(k)}$ from the last row, we find that

$$\langle y_k, y_j \rangle = R_{jk}.$$

Next we show that $\langle y_k, y_k \rangle = R_{kk}$. Since $\langle \omega_k, y_j \rangle = 0$ for $\beta(k) \leqslant j < k$, it follows from (3.3) that $\langle \omega_k, y_k \rangle = \Delta_k$. Then from (3.3) and the linearity of the determinant in the bottom row,we have

$$\langle y_k, y_k \rangle = \Delta_k^2 - \frac{1}{\det R(\beta(k),\dots,k-1)}$$

$$\times \det \begin{bmatrix} R_{\beta(k),\beta(k)} & \cdots & R_{\beta(k),k-1} & R_{\beta(k),k} \\ \vdots & & \vdots & \vdots \\ R_{k-1,\beta(k)} & \cdots & R_{k-1,k-1} & R_{k-1,k} \\ R_{k,\beta(k)} & \cdots & R_{k,k-1} & 0 \end{bmatrix}$$

$$= \Delta_k^2 - \Delta_k^2 + \frac{1}{\det R(\beta(k),\dots,k-1)}$$

$$\times \det \begin{bmatrix} R_{\beta(k),\beta(k)} & \cdots & R_{\beta(k),k-1} & R_{\beta(k),k} \\ \vdots & & \vdots & \vdots \\ R_{k-1,\beta(k)} & \cdots & R_{k-1,k-1} & R_{k-1,k} \\ 0 & \cdots & 0 & R_{kk} \end{bmatrix}$$

$$= R_{kk}.$$

Thus $\{y_k\}_{k=1}^n$ is an admissible sequence.

Now suppose that $\{y_k\}_{k=1}^n$ is an admissible m-band sequence, and define $\{\omega_k\}_{k=1}^n$ by applying the Gram-Schmidt process to $\{y_k\}$. By (3.2), Proposition 2.4, (2.3)–(2.7), and (3.1) we have

$$y_1 = \sqrt{R_{11}}\,\omega_1 \tag{3.4}$$

and

$$y_k = \operatorname{proj}(y_k, \operatorname{sp}\langle y_{\beta(k)},\dots,y_{k-1}\rangle) + \Delta_k\omega_k \qquad (2 \leqslant k \leqslant n). \tag{3.5}$$

However, it is known [7, p. 15] that

$$\operatorname{proj}(y_k, \operatorname{sp}\langle y_{\beta(k)}, \dots, y_{k-1}\rangle)$$
$$= \frac{-1}{\det \Gamma(y_{\beta(k)}, \dots, y_{k-1})}$$
$$\times \det \begin{bmatrix} \langle y_{\beta(k)}, y_{\beta(k)}\rangle & \cdots & \langle y_{k-1}, y_{\beta(k)}\rangle & \langle y_k, y_{\beta(k)}\rangle \\ \vdots & & \vdots & \vdots \\ \langle y_{\beta(k)}, y_{k-1}\rangle & & \langle y_{k-1}, y_{k-1}\rangle & \langle y_k, y_{k-1}\rangle \\ y_{\beta(k)} & \cdots & y_{k-1} & 0 \end{bmatrix}. \quad (3.6)$$

Since $\{y_k\}_{k=1}^n$ is admissible, it follows from (3.4)–(3.6) that $y_1, \dots, y_n$ satisfy (3.2) and (3.3). This completes the proof of the theorem. ∎

THEOREM 3.3. *Let R be an $n \times n$ band matrix with bandwidth m such that $R(j, \dots, j+m)$ is positive definite for $1 \leqslant j \leqslant n-m$. Then every solution of the maximum distance problem for R is an m-band sequence and hence is given by* (3.2) *and* (3.3) *for some orthonormal set* $\{\omega_k\}_{k=1}^n$.

Proof. By Theorem 3.2 there is an m-band sequence that is admissible for R. Hence, by Theorem 3.1, $d_k = \Delta_k$ for $2 \leqslant k \leqslant n$. Therefore if $y_1, \dots, y_n$ is a solution of the maximum distance problem, then

$$\operatorname{dist}(y_k, \operatorname{sp}\langle y_1, \dots, y_{k-1}\rangle) = d_k = \Delta_k = \operatorname{dist}(y_k, \operatorname{sp}\langle y_{\beta(k)}, \dots, y_{k-1}\rangle).$$

By Proposition 2.4, $\{y_k\}_{k=1}^n$ is an m-band sequence. ∎

There is another description of the solutions of the maximum distance problem. (See [5] and [6, Theorem 1].) We present it in the next theorem.

THEOREM 3.4. *Let R be an $n \times n$ band matrix with bandwidth m such that $R(j, \dots, j+m)$ is positive definite for $1 \leqslant j \leqslant n-m$. Let W be the lower triangular band matrix with bandwidth m whose entries within the lower half band are defined by*

$$R(j, \dots, \gamma(j)) \begin{bmatrix} w_{jj} \\ w_{j+1,j} \\ \vdots \\ w_{\gamma(j),j} \end{bmatrix} = \begin{bmatrix} w_{jj}^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (1 \leqslant j \leqslant n) \qquad (3.7)$$

with $w_{jj} > 0$ for $1 \leqslant j \leqslant n$. Then W is invertible and the columns of W^{-1} form a solution of the maximum distance problem for R.

Proof. Since the matrices $R(j,\dots,\gamma(j))$ are positive definite, the equations in (3.7) have unique solutions with $w_{jj} > 0$ for $1 \leqslant j \leqslant n$. Let W be the lower triangular band matrix whose entries w_{kj} in the lower half band are given by (3.7). Then W is invertible. Also, the entries in the product RW inside the band are zero below the diagonal and coincide with the entries of W^{*-1} on the diagonal.

Let B be the matrix that coincides with RW below the band and has zeros inside and above the band. Then BW^{-1} also has zeros inside and above the band, because W^{-1} is lower triangular. If we define

$$F = -BW^{-1} + R - (BW^{-1})^*,$$

then F is clearly a self-adjoint matrix that agrees with R in the band. Let us examine the product

$$FW = -B + RW - (BW^{-1})^* W.$$

The definition of B shows that $-B + RW$ is upper triangular and agrees with W^{*-1} on the diagonal. It is easy to see that $(BW^{-1})^* W$ is upper triangular with zeros on the diagonal. Thus FW is upper triangular and agrees with W^{-1} on the diagonal. Since W^* is upper triangular, it follows that W^*FW is a self-adjoint upper-triangular matrix with diagonal entries equal to 1, so $W^*FW = I$. Letting $X = W^{-1}$, we have

$$F = X^*X = \Gamma(x_1,\dots,x_n),$$

where $x_1,\dots,x_n$ are the columns of X. Thus the columns of X form an admissible set of vectors for the maximum distance problem for R. Furthermore, the Gram matrix of this set of vectors is invertible, and its inverse is the band matrix WW^*. By Proposition 2.1, $\{x_k\}_{k=1}^n$ is an m-band sequence. Hence it is a solution of the maximum distance problem for R, by Theorem 3.1. ∎

4. EQUIVALENT m-BAND SEQUENCES AND EXAMPLES

From Proposition 2.1 it follows that if $\{y_k\}_{k=1}^n$ is an m-band sequence in $\mathbf{C}^N$ and U is an $N \times N$ unitary matrix, then $\{Uy_k\}_{k=1}^n$ is an m-band sequence. This motivates the following definition.

DEFINITION. Two linearly independent sequences $\{y_k\}_{k=1}^n$ and $\{z_k\}_{k=1}^n$ in $\mathbf{C}^N$ are *equivalent* if there is an $N \times N$ unitary matrix U such that $z_k = Uy_k$ for $1 \leqslant k \leqslant n$.

This defines an equivalence relation among linearly independent sequences in $\mathbf{C}^N$. The equivalence classes are described in the next result.

PROPOSITION 4.1. *Let $\{y_k\}_{k=1}^n$ and $\{z_k\}_{k=1}^n$ be linearly independent sequences in $\mathbf{C}^N$. The following statements are equivalent:*

(1) *$\{y_k\}_{k=1}^n$ and $\{z_k\}_{k=1}^n$ are equivalent.*
(2) *$\Gamma(y_1,\dots,y_n) = \Gamma(z_1,\dots,z_n)$.*
(3) *$\{y_k\}_{k=1}^n$ and $\{z_k\}_{k=1}^n$ have the same orthonormalization matrix.*

Proof. The proofs that (1) is equivalent to (2) and that (3) implies (1) are easy. Suppose that (1) holds, and let $Y = [y_1 \ \cdots \ y_n]$ and $Z = [z_1 \ \cdots \ z_n]$. Then there is a unitary $N \times N$ matrix U such that $UY = Z$. Let $\{\omega_k\}_{k=1}^n$ be the sequence obtained by applying the Gram-Schmidt process to $y_1,\dots,y_n$, and let A be the associated orthonormalization matrix. Let $\{\omega'_k\}_{k=1}^n$ and B be the corresponding objects for $\{z_k\}_{k=1}^n$. Since $Uy_j = z_j$ for $1 \leqslant j \leqslant n$, it follows from Equations (2.3)–(2.7) for the Gram-Schmidt process that $U\omega_j = \omega'_j$ for $1 \leqslant j \leqslant n$. Thus $UYA = ZB = UYB$, and so $A = B$. Thus (1) implies (3). ∎

For m-band sequences only the entries in the band of the Gram matrix are needed to determine the equivalence class, as we prove next.

PROPOSITION 4.2. *Let $0 \leqslant m < n$. The following are equivalent for two m-band sequences $\{y_k\}_{k=1}^n$ and $\{z_k\}_{k=1}^n$ in C^N:*

(1) *The sequences are equivalent.*

(2) *The sequences are solutions of the maximum distance problem for the same band matrix R with bandwidth m.*

(3) *The bands of bandwidth m of $\Gamma(y_1,\dots,y_n)$ and $\Gamma(z_1,\dots,z_n)$ are identical.*

Proof. The three implications $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ are provided by Proposition 4.1, Theorem 3.1, and Theorem 3.3, respectively. ∎

From Proposition 4.2 it follows that the equivalence classes of m-band sequences of length n in $\mathbf{C}^N$ are in one-to-one correspondence with $n \times n$ band matrices R with bandwidth m having the property that $R(j,\dots,j+m)$ is positive definite for $1 \leqslant j \leqslant n-m$. Moreover, given such a band matrix R, we can find a representative of the equivalence class determined by R, either by using the recursion formula in (3.2) and (3.3) or by taking the columns of the inverse of the lower triangular band matrix W defined by the systems of equations in (3.7).

EXAMPLE 4.1. Let us determine representatives of all equivalence classes of 1-band sequences. Let

$$R = \begin{bmatrix} 1 & \bar{b}_1 & 0 & \cdots & 0 & 0 \\ b_1 & 1 & \bar{b}_2 & \cdots & 0 & 0 \\ 0 & b_2 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 1 & \bar{b}_{n-1} \\ 0 & 0 & 0 & \cdots & b_{n-1} & 1 \end{bmatrix}$$

where $|b_j| < 1$ for $1 \leqslant j \leqslant n-1$. Here R has been normalized so that its diagonal entries are 1, but otherwise R is arbitrary band matrix with bandwidth 1 such that $R(j,\dots,j+1)$ is positive definite for $1 \leqslant j \leqslant n-1$. The lower triangular band matrix W defined by (3.7) is given by

$$W = \begin{bmatrix} c_1^{-1} & 0 & \cdots & 0 & 0 \\ -b_1c_1^{-1} & c_2^{-1} & \cdots & 0 & 0 \\ 0 & -b_2c_2^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -b_{n-1}c_{n-1}^{-1} & 1 \end{bmatrix}$$

where $c_j = (1-|b_j|^2)^{1/2}$. The inverse X of W is given by

$$X = \begin{bmatrix} c_1 & 0 & \cdots & 0 & 0 \\ b_1c_2 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ b_1\cdots b_{n-2}c_{n-1} & b_2\cdots b_{n-2}c_{n-1} & \cdots & c_{n-1} & 0 \\ b_1\cdots b_{n-1} & b_2\cdots b_{n-1} & \cdots & b_{n-1} & 1 \end{bmatrix}.$$

The columns of X have norm 1 and form a 1-band sequence. Every equivalence class is represented in this way. The Gram matrix of this 1-band sequence is X^*X, which equals

$$\begin{bmatrix} 1 & \bar{b}_1 & \cdots & \bar{b}_1\cdots\bar{b}_{n-2} & \bar{b}_1\cdots\bar{b}_{n-1} \\ b_1 & 1 & \cdots & \bar{b}_2\cdots\bar{b}_{n-2} & \bar{b}_2\cdots\bar{b}_{n-1} \\ b_1b_2 & b_2 & & \vdots & \vdots \\ \vdots & \vdots & & 1 & \bar{b}_{n-1} \\ b_1\cdots b_{n-1} & b_2\cdots b_{n-1} & \cdots & b_{n-1} & 1 \end{bmatrix}$$

If we take $b_1 = \cdots = b_{n-1} = \alpha$, where $|\alpha| < 1$, we find that

$$W = c^{-1}\cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha & c \end{bmatrix}$$

and

$$X = c\cdot \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha & 1 & \cdots & 0 & 0 \\ \alpha^2 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha^{n-2} & \alpha^{n-3} & \cdots & 1 & 0 \\ \alpha^{n-1}c^{-1} & \alpha^{n-2}c^{-1} & \cdots & \alpha c^{-1} & c^{-1} \end{bmatrix},$$

where $c = (1-|\alpha|^2)^{1/2}$. Thus the band extension of R is given by

$$\begin{bmatrix} 1 & \bar{\alpha} & \bar{\alpha}^2 & \cdots & \bar{\alpha}^{n-2} & \bar{\alpha}^{n-1} \\ \alpha & 1 & \bar{\alpha} & \cdots & \bar{\alpha}^{n-3} & \bar{\alpha}^{n-2} \\ a^2 & \alpha & 1 & \cdots & \bar{\alpha}^{n-4} & \bar{\alpha}^{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha^{n-1} & \alpha^{n-2} & \alpha^{n-3} & \cdots & \alpha & 1 \end{bmatrix}.$$

5. RELATED PROBLEMS

Closely related to the maximum distance problem is the maximum volume problem. Let $\{a_{jk} : |j-k| \leqslant m\}$ be a given set of complex numbers with

$a_{kj} = \bar{a}_{jk}$. For any $x_1, \ldots, x_n$ in $\mathbf{C}^n$ we denote by $V(x_1, \ldots, x_n)$ the volume of the parallelepiped determined by $x_1, \ldots, x_n$. We define

$$V = \sup V(x_1, \ldots, x_n),$$

where the supremum is taken over all linearly independent sets of vectors $x_1, \ldots, x_n$ that satisfy the constraints

$$\langle x_k, x_j \rangle = a_{jk} \qquad (|j-k| \leqslant m). \tag{5.1}$$

The *maximum volume problem* is to describe all linearly independent sets of vectors $x_1, \ldots, x_n$, subject to the constraints in (5.1), such that

$$V(x_1, \ldots, x_n) = V.$$

THEOREM 5.1. *The maximum volume problem with constraints in* (5.1) *has a solution if and only if all of the matrices*

$$(a_{rs})_{r,s=j}^{j+m} \qquad (1 \leqslant j \leqslant n-m)$$

are positive definite. In that case, the following are equivalent for a basis $\{x_k\}_{k=1}^n$ *of* $\mathbf{C}^n$ *that satisfies the constraints in* (5.1):

(1) $\{x_k\}_{k=1}^n$ *is a solution of the maximum volume problem.*
(2) $\{x_k\}_{k=1}^n$ *is a solution of the maximum distance problem.*
(3) $\{x_k\}_{k=1}^n$ *is an m-band sequence.*

Proof. Let R be the band matrix whose entries in the band are given by $R_{jk} = a_{jk}$ for $|j-k| \leqslant m$. Observe that a linearly independent set of vectors $x_1, \ldots, x_n$ is a solution of the maximum volume problem if and only if $\Gamma(x_1, \ldots, x_n)$ is a positive definite matrix that agrees with R in the band and

$$V(x_1, \ldots, x_n) = V.$$

But it is well known [7, p. 15] that

$$\begin{aligned}
[V(x_1, \ldots, x_n)]^2 &= \det \Gamma(x_1, \ldots, x_n) \\
&= \|x_1\|^2 \prod_{k=2}^{n} \frac{\det \Gamma(x_1, \ldots, x_k)}{\det \Gamma(x_1, \ldots, x_{k-1})} \\
&= \|x_1\|^2 \prod_{k=2}^{n} \operatorname{dist}(x_k, \operatorname{sp}\langle x_1, \ldots, x_{k-1} \rangle)^2.
\end{aligned}$$

Since $\|x_1\| = a_{11}$, which is fixed, it follows that $V(x_1,\dots,x_n) = V$ if and only if $\operatorname{dist}(x_k, \operatorname{sp}\langle x_1,\dots,x_n\rangle) = d_k$ for $1 \leqslant k \leqslant n$. This proves the equivalence of (1) and (2). The equivalence of (2) and (3) follows from Theorems 3.1 and 3.3. ■

There is an interesting special case of the maximum distance problem. Suppose that $\{x_k\}_{k=1}^{r}$ and $\{x_k\}_{k=r+1}^{2r}$ are orthonormal sets in $\mathbf{C}^{2r}$. Then the Gram matrix of $\{x_k\}_{k=1}^{2r}$ has the form

$$\begin{bmatrix} I & B \\ B^* & I \end{bmatrix},$$

where B is an $r \times r$ matrix and I is the $r \times r$ identity matrix. Let m be an integer such that $1 \leqslant m \leqslant 2r-1$, and let $S = \{(j,k) : \max(1, r-m+1) \leqslant j \leqslant r,\ 1 \leqslant k \leqslant \min(r, j+m-r)\}$. Given a set $\{a_{jk} : (j,k) \in S\}$ of complex numbers, we would like to describe all linearly independent sets of vectors $\{x_k\}_{k=1}^{2r}$ such that $\{x_k\}_{k=1}^{r}$ and $\{x_k\}_{k=r+1}^{2r}$ are orthonormal sets, the constraints

$$\langle x_{k+r}, x_j \rangle = a_{jk} \qquad [(j,k) \in S] \tag{5.2}$$

are satisfied, and

$$\operatorname{dist}(x_k, \operatorname{sp}\langle x_1,\dots,x_{k-1}\rangle) = d_k \qquad \text{for} \quad 2 \leqslant k \leqslant 2r.$$

We shall refer to this problem as the *special maximum distance problem*.

Observe that a linearly independent set of vectors $\{x_k\}_{k=1}^{2r}$, with $\{x_k\}_{k=1}^{r}$ and $\{x_k\}_{k=r+1}^{2r}$ orthonormal, satisfies the constraints in (5.2) if and only if $\Gamma(x_1,\dots,x_{2r})$ is a positive definite matrix that agrees in the band with the matrix R defined by

$$R = \begin{bmatrix} I & A \\ A^* & I \end{bmatrix} \tag{5.3}$$

where I is the $r \times r$ identity matrix and

$$A_{jk} = \begin{cases} a_{jk} & \text{for} \quad (j,k) \in S, \\ 0 & \text{otherwise}. \end{cases} \tag{5.4}$$

For any matrix M we denote by $\|M\|$ the norm of M as an operator between Euclidean spaces.

THEOREM 5.2. *The special maximum distance problem with constraints in* (5.2) *has a solution if and only if* $\|M\| < 1$ *for every matrix* M *of the form*

$$M = \left(A_{pq}\right)_{j \leqslant p \leqslant r, 1 \leqslant q \leqslant \min(r, j+m-r)} \qquad \left[\max(1, r-m+1) \leqslant j \leqslant r\right], \tag{5.5}$$

where A_{pq} *is defined by* (5.4). *In this case let* $x_1, \dots, x_n$ *be a solution of the special maximum distance problem, and let* $y_1, \dots, y_n$ *be any admissible set of vectors for the special maximum distance problem, where* $n = 2r$. *If the corresponding Gram matrices are written in the form*

$$\Gamma(x_1, \dots, x_n) = \begin{bmatrix} I & B \\ B^* & I \end{bmatrix} \tag{5.6}$$

and

$$\Gamma(y_1, \dots, y_n) = \begin{bmatrix} I & C \\ C^* & I \end{bmatrix}, \tag{5.7}$$

then

$$\det(I - B^*B) \geqslant \det(I - C^*C)$$

with equality if and only if $C = B$.

Proof. Each of the matrices $R(j, \dots, j+m)$ is of the form

$$\begin{bmatrix} I_1 & M \\ M^* & I_2 \end{bmatrix} \tag{5.8}$$

where M is given by (5.5), and I_1 and I_2 are identity matrices. We shall show that the matrix in (5.8) is positive definite if and only if $\|M\| < 1$. First we write

$$\begin{bmatrix} I_1 & M \\ M^* & I_2 \end{bmatrix} = S^* \begin{bmatrix} I_1 & 0 \\ 0 & I_2 - M^*M \end{bmatrix} S$$

where

$$S = \begin{bmatrix} I_1 & M \\ 0 & I_2 \end{bmatrix}.$$

Since S is invertible, it follows that the matrix in (5.8) is positive definite if and only if $I_2 - M^*M$ is positive definite, and the latter condition is equivalent to $\|M\| < 1$. In view of Theorem 3.4, this proves the first statement of Theorem 5.2. For the second statement we use the equivalence of the maximum distance problem and the maximum volume problem, and observe that the square of the volume of the parallelepiped determined by $y_1, \dots, y_n$ is equal to the determinant of the matrix in (5.7), which equals $\det(I - C^*C)$. If $\det(I - B^*B) = \det(I - C^*C)$, then $\{y_k\}_{k=1}^n$ is a solution of the maximum volume problem and hence the maximum distance problem. Thus $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$ are equivalent m-band sequences, so $\Gamma(y_1, \dots, y_n) = \Gamma(x_1, \dots, x_n)$, i.e., $C = B$. ∎

Let A be an $r \times r$ matrix in the form of (5.4), and assume that $\|M\| < 1$ for each matrix in (5.5). By Theorem 5.2 the special maximum distance problem has a solution. Let $\{x_k\}_{k=1}^{2r}$ have the property that $\{x_k\}_{k=1}^{r}$ and $\{x_k\}_{k=r+1}^{2r}$ are orthonormal sets. Then the orthonormalization matrix for $\{x_k\}_{k=1}^{2r}$ has the form

$$\begin{bmatrix} I & D \\ 0 & I \end{bmatrix}, \tag{5.9}$$

and the Gram matrix has the form

$$\begin{bmatrix} I & B \\ B^* & I \end{bmatrix} \tag{5.10}$$

According to Theorems 3.1 and 3.3, $\{x_k\}_{k=1}^{2r}$ is a solution of the maximum distance problem if and only if it is an m-band sequence. This is equivalent to the matrix in (5.9) being a band matrix with bandwidth m and the matrix in (5.10) having a banded inverse with bandwidth m. The next theorem characterizes this matrix B in the case $m = r$, which means that A is lower triangular. The theorem can easily be extended to cover other values of m.

THEOREM 5.3. *Let $A = (A_{jk})$ be an $r \times r$ lower triangular matrix, and assume that $\|M\| < 1$ for every matrix M of the form*

$$(A_{pq})_{j \leqslant p \leqslant r,\, 1 \leqslant q \leqslant \min(r,\, j+m-r)} \qquad [\max(1, r-m+1) \leqslant j \leqslant r].$$

Then the matrix in (5.10) is the Gram matrix of a solution of the special maximum distance problem if and only if $\|B\| < 1$ and $B(I - B^*B)^{-1}$ is lower triangular.

Proof. The matrix in (5.10) is positive definite if and only if $\|B\| < 1$. According to our comments preceding the theorem, the matrix in (5.10) is the Gram matrix of a solution if and only if its inverse is a band matrix with bandwidth r. But from the factorization

$$\begin{bmatrix} I & B \\ B^* & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I - B^*B \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$$

we have

$$\begin{bmatrix} I & B \\ B^* & I \end{bmatrix}^{-1} = \begin{bmatrix} I + B(I - B^*B)^{-1}B^* & -B(I - B^*B)^{-1} \\ -(I - B^*B)^{-1}B^* & (I - B^*B)^{-1} \end{bmatrix}.$$

From this the result follows immediately. ■

Theorem 5.3 gives a unique way to complete a lower triangular matrix A to a matrix B with norm less than 1. A similar result appears in [2, Theorem III.9]. Problems of this type (without uniqueness) are treated in [4].

REFERENCES

1 E. Asplund, Inverses of matrices $\{a_{ij}\}$ which satisfy $a_{ij} = 0$ for $j > i + p$, *Math. Scand.* 7:57–60 (1959).
2 J. Ball and I. Gohberg, Shift invariant subspaces, factorization, and interpolation for matrices. I. The canonical case, *Linear Algebra Appl.*, 74:87–150 (1986).
3 J. Burg, Maximum entropy spectral analysis, Ph.D. Dissertation, Stanford Univ., Stanford, Calif., 1975.
4 C. Davis, W. M. Kahan, and H. F. Weinberger, Norm-preserving dilations and their applications to optimal error bounds, *SIAM J. Numer. Anal.* 19:445–469 (1982).
5 H. Dym and I. Gohberg, Extensions of band matrices with band inverses, *Linear Algebra Appl.* 36:1–24 (1981).
6 R. L. Ellis, I. Gohberg, and D. C. Lay, Band extensions, maximum entropy and the permanence principle, in *Maximum Entropy and Bayesian Methods in Applied Statistics* (J. Justice, Ed.), Cambridge U.P., 1986.
7 I. Gohberg and S. Goldberg, *Basic Operator Theory*, Birkhäuser, Boston, 1981.

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